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# Nonlocal symmetries and a Darboux transformation for the Camassa–Holm equation

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**Abstract**
We announce two new structures associated with the Camassa–Holm (CH) equation: a Lie algebra of *nonlocal* symmetries, and a Darboux transformation for this important equation, which we construct using only our symmetries. We also extend our results to the associated Camassa–Holm equation introduced by J Schiff (1998 *Physica* D **121** 24–43).

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## 1. Introduction

It was shown in [20] that the Camassa–Holm (CH) equation [3, 4],

$$2u_x u_{xx} + u u_{xxx} = u_t - u_{xxt} + 3u_x u, \tag{1}$$

is the integrability condition of an overdetermined $sl(2, \mathbf{R})$-valued linear problem, and that it possesses a nonlocal symmetry essentially arising from it. Nonlocal symmetries are interesting, because it has been observed that they carry information about the existence of linearizing and Bäcklund/Darboux transformations [2, 13, 14, 21, 24, 25], and also because they allow us to construct explicit non-trivial solutions [8, 15, 16, 21, 24], a topic of importance for numerical methods, for instance. It is then natural to continue their investigation in the ubiquitous case of equation (1). In this work, we identify a whole Lie algebra of $(x, t)$-independent nonlocal symmetries for CH and, motivated by previous work [21, 24], we construct a Darboux transformation for this equation with the help of one of our symmetries. This is important, since to the best of our knowledge the direct construction of such a transformation (without using hodographic transformations and/or transformations of Schrödinger operators as, for instance, in [1, 23]) has been an open problem since Camassa and Holm's paper [3] in 1993.

  

For completeness, we also identify a Lie algebra of nonlocal symmetries of, and construct a Darboux transformation for, the associated Camassa–Holm equation (ACH) introduced by Schiff in [23]. In this case, our transformation coincides with that obtained in [23] by using loop groups techniques.

We omit most technical details in this paper. They, and further results along these lines, appear elsewhere [10].

Familiarity with the theory of classical and generalized symmetries, as it appears, for instance in [14, 17], is assumed throughout.

*Notational conventions.* Hereafter independent variables are denoted by $x^i, i = 1, 2, \ldots, n$, and dependent variables by $u^\alpha, \alpha = 1, 2, \ldots, m$. Also, partial derivatives with respect to $x^i$ are indicated by sub-indices, $D_i$ stands for the total derivative with respect to $x^i$,

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{\#J \geqslant 0} u^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J}, \tag{2}$$

(in which the *unordered* $k$-tuple $J = (j_1, \ldots, j_k), 0 \leqslant j_1, j_2, \ldots, j_k \leqslant n$ indicates a multi-index of order $\#J = k, u^\alpha_{Ji} = \partial u^\alpha_J / \partial x^i$), and $D_J$ indicates the composition $D_J = D_{j_1} D_{j_2} \ldots D_{j_k}$.

## 2. Nonlocal symmetries of partial differential equations

Nonlocal symmetries have been studied rigorously by Vinogradov and Krasil'shchik, see [13, 14, 25]. Here we give only a short summary of their theory; the reader is referred to the three papers just cited for full details (see also [21, 22] for recent elementary discussions).

We begin with an example appearing already in [25].

**Example 1.** We consider Burgers' equation $u_t = u_{xx} + uu_x$. The expression

$$G = (2S_x - uS)\, \mathrm{e}\left(-\frac{1}{2}\int u \,\mathrm{d}x\right), \tag{3}$$

in which $S$ is any function such that $S_t = S_{xx}$, satisfies the following condition: if $u$ is a solution to Burgers' equation, the 'deformation' $u + \tau G$ is also a solution to first order in $\tau$, that is, $G$ formally satisfies the linearized Burgers equation. One way to make this observation rigorous would be to consider an extra dependent variable $\gamma^1$ such that $\gamma^1_x = u$ and $\gamma^1_t = u_x + (1/2)u^2$. Then, we can write (3) as $G = (2S_x - uS)\exp\left(-\frac{1}{2}\gamma^1\right)$, so that $G$ becomes 'local' and could perhaps be considered as the characteristic of a local symmetry for the 'augmented' system

$$u_t = u_{xx} + uu_x, \qquad \gamma^1_x = u, \qquad \gamma^1_t = u_x + (1/2)u^2. \tag{4}$$

But then, in order to formalize this idea, we also need to consider the infinitesimal variation of $\gamma^1$ as $u$ changes to $u + \tau G$. These remarks motivate the following two definitions.

**Definition 1.** *Let $N$ be a non-zero integer. An $N$-dimensional covering $\pi$ of a (system of) partial differential equation(s) $\Xi_a = 0, a = 1, \ldots, k$, is a pair*

$$\pi = (\{\gamma^b : b = 1, \ldots, N\}; \{X_{ib} : b = 1, \ldots, N; i = 1, \ldots, n\}) \tag{5}$$

*of variables $\gamma^b$ and smooth functions $X_{ib}$ depending on $x^i, u^\alpha, \gamma^b$ and a finite number of partial derivatives of $u^\alpha$, such that the equations,*

$$\frac{\partial \gamma^b}{\partial x^i} = X_{ib}, \tag{6}$$

*are compatible whenever $u^\alpha(x^i)$ is a solution to $\Xi_a = 0$.*

We usually write $\pi = (\gamma^b; X_{ib})$ instead of (5). Generalizing example 1, we consider the variables $\gamma_b$ as new dependent variables, the 'nonlocal variables' of the theory. Equation (6) then states how they relate to the original variables $u^\alpha$. In example 1, setting $x_1 = x$ and $x_2 = t$, we have $N = 1$, $X_{11} = u$, $X_{21} = u_x + (1/2)u^2$, and the last two equations of (4) correspond to equation (6).

We define nonlocal symmetries as follows (compare [14, pp 249–50]).

**Definition 2.** *Let $\Xi_a = 0$, $a = 1, \ldots, k$, be a system of partial differential equations, and let $\pi = (\gamma^b; X_{ib})$ be a covering of $\Xi_a = 0$. A nonlocal $\pi$-symmetry of $\Xi_a = 0$ is a generalized symmetry,*

$$X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_\alpha \phi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_b \varphi^b \frac{\partial}{\partial \gamma^b},$$

*of the augmented system*

$$\Xi_a = 0, \qquad \frac{\partial \gamma^b}{\partial x^i} = X_{ib}. \tag{7}$$

Thus, in order to find nonlocal $\pi$-symmetries, we proceed exactly as in the local case considered, for instance, in Olver's treatise [17, chapter 5]: we need to check the conditions [14], [17, p 290]

$$pr X(\Xi_a) = 0, \qquad \text{and} \qquad pr X\left(\frac{\partial \gamma^b}{\partial x^i} - X_{ib}\right) = 0, \tag{8}$$

in which

$$pr X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha, J} \phi_J^\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{b,J} \varphi_J^b \frac{\partial}{\partial \gamma_J^b}$$

and

$$\phi_J^\alpha = D_J\left(\phi^\alpha - \sum_i \xi^i u_i^\alpha\right) + \sum_i \xi^i u_{Ji}^\alpha, \qquad \text{and} \qquad \varphi_J^b = D_J\left(\varphi^b - \sum_i \xi^i \gamma_i^b\right) + \sum_i \xi^i \gamma_{Ji}^b.$$

Now, as explained in [17, p 291], it is enough to consider 'evolutionary' vector fields $X$ of the form

$$X = \sum_{\alpha=1}^m G^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{b=1}^N H^b \frac{\partial}{\partial \gamma^b}, \tag{9}$$

and it is well known (see, for instance, [17, p 307] and also [10, 21] for elementary discussions within the present framework) that in this case the generalized symmetry conditions (8) say that the infinitesimal deformation $u^\alpha \mapsto u^\alpha + \tau G^\alpha$ satisfies the system of equations $\Xi_a = 0$ to first order in the deformation parameter $\tau$, and that the infinitesimal deformation $\gamma^b \mapsto \gamma^b + \tau H^b$ satisfies the compatible system (6) to first order in $\tau$. Thus, we have completely formalized our example 1.

**Corollary 1.** *If $u_0^\alpha(x^i)$ and $\gamma_0^b(x^i)$ are solutions to the augmented system (7), the solution to the Cauchy problem*

$$\frac{\partial u^\alpha}{\partial \tau} = G^\alpha, \qquad \frac{\partial \gamma^b}{\partial \tau} = H^b,$$
$$u^\alpha(x^i, 0) = u_0^\alpha(x^i), \qquad \gamma_b(x^i, 0) = \gamma_0^b(x^i),$$

*is a one-parameter family of solutions to the augmented system (7). In particular, nonlocal $\pi$-symmetries send solutions to the system $\Xi_a = 0$ to solutions of the same system.*

We finish this section with a computational note: since we are allowed to replace all derivatives of $\gamma_b$ appearing in (9) by means of equations (6), see [17, p 292], we can assume without loss of generality that the coefficients $G^\alpha$ and $H_b$ of the vector field (9) depend *only* on $x^i$, $u^\alpha$, finite numbers of derivatives of $u^\alpha$, and the new variables $\gamma^b$. This simplification is crucial for obtaining the classification results we present next.

## 3. Nonlocal symmetries for the CH and ACH equations

We write the CH equation (1) as a system of equations for two dependent variables $m$ and $u$:

$$m = u_{xx} - u, \qquad m_t = -m_x u - 2mu_x. \tag{10}$$

Our first theorem is [20]

**Theorem 1.** *The system of first-order equations,*

$$\gamma_x = m - \frac{1}{2\lambda}\gamma^2 + \frac{1}{2}\lambda, \qquad \gamma_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right)_x, \tag{11}$$

*is completely integrable on solutions to (10) and therefore it determines a pseudo-potential $\gamma$ for the CH equation. Moreover, the following two systems of equations are compatible whenever $u(x,t)$ and $m(x,t)$ are solutions to (10):*

$$\delta_x = \gamma, \qquad \delta_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right) \tag{12}$$

*and*

$$\beta_x = m\,\mathrm{e}^{\delta/\lambda}, \qquad \beta_t = \mathrm{e}^{\delta/\lambda}\left(-\frac{1}{2}\gamma^2 + \frac{1}{2}\lambda^2 - um\right). \tag{13}$$

Theorem 1 can be checked by direct computations. The system (11) is the 'Riccati form' of the linear problem associated with the CH equation (10) which we mentioned in section 1 (see [20] and the classical paper by Chen [5]); equations (12) and (13) determine (sequences of) conservation laws, see [20, 10].

The systems of equations (11)–(13) allow us to define a three-dimensional covering $\pi$ of the CH equation, the nonlocal variables are $\gamma$, $\delta$ and $\beta$. We now classify nonlocal $\pi$-symmetries.

**Theorem 2.** *The first-order generalized symmetries of the augmented CH system (10)–(13), represented by vector fields (9), with $G^\alpha$ and $H_b$ being functions of $m$, $u$, $\gamma$, $\delta$, $\beta$, $m_x$, $u_x$ and $u_t$ only, are linear combinations of*

$$V_1 = (2mu_x + um_x)\frac{\partial}{\partial m} - u_t\frac{\partial}{\partial u} + \left(\frac{\lambda^2}{2} - \frac{\lambda u}{2} + um - \frac{\gamma^2}{2} - \frac{u\gamma^2}{2\lambda} + \gamma u_x\right)\frac{\partial}{\partial \gamma}$$
$$+ (\lambda\gamma + u\gamma - \lambda u_x)\frac{\partial}{\partial \delta} - \frac{1}{2}\mathrm{e}^{\delta/\lambda}(\lambda^2 - 2um - \gamma^2)\frac{\partial}{\partial \beta}, \tag{14}$$

$$V_2 = m_x\frac{\partial}{\partial m} + u_x\frac{\partial}{\partial u} + \left(\frac{\lambda}{2} + m - \frac{\gamma^2}{2\lambda}\right)\frac{\partial}{\partial \gamma} + \gamma\frac{\partial}{\partial \delta} + \mathrm{e}^{\delta/\lambda}m\frac{\partial}{\partial \beta}, \tag{15}$$

$$V_3 = \frac{\partial}{\partial \delta} + \frac{\beta}{\lambda}\frac{\partial}{\partial \beta}, \tag{16}$$

$$V_4 = \frac{\partial}{\partial \beta}, \tag{17}$$

$$V_5 = e^{\delta/\lambda}\left(\frac{2m\gamma}{\lambda} + m_x\right)\frac{\partial}{\partial m} + e^{\delta/\lambda}\gamma\frac{\partial}{\partial u} + e^{\delta/\lambda}m\frac{\partial}{\partial \gamma} + \beta\frac{\partial}{\partial \delta} + \left(e^{2\delta/\lambda}m + \frac{\beta^2}{2\lambda}\right)\frac{\partial}{\partial \beta}. \tag{18}$$

*Consequently, these vector fields are nonlocal $\pi$-symmetries of the CH equation.*

Theorem 2 is proven via extensive symbolic computations carried out with the help of *Mathematica* software written by the authors [9].

**Corollary 2.** *The five nonlocal $\pi$-symmetries (14)–(18) generate a Lie algebra with the commutator table:*

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ |
|---|---|---|---|---|---|
| $V_1$ | | | | | |
| $V_2$ | | | | | |
| $V_3$ | | | | $-\frac{1}{\lambda}V_4$ | $\frac{1}{\lambda}V_5$ |
| $V_4$ | | | $\frac{1}{\lambda}V_4$ | | $V_3$ |
| $V_5$ | | | $-\frac{1}{\lambda}V_5$ | $-V_3$ | |

.

We stress the fact that this Lie algebra exists because we work on a fixed covering of the CH equation: we cannot expect the 'space of all nonlocal symmetries' of a given equation to possess a Lie algebra structure; see [13] and also [18, 19].

We now consider the ACH equation introduced by Schiff in [23] and further discussed in [7, 11, 12]. Set

$$p = \sqrt{2m}, \qquad dy = p\ dx - pu\ dt \qquad \text{and} \qquad dT = dt, \tag{19}$$

and replace in equation (10). We find Schiff's *ACH equation*

$$p_T = -p^2 u_y, \qquad u = -\frac{p^2}{2} - \left(\frac{p_T}{p}\right)_y p. \tag{20}$$

We have three results analogous to those obtained for CH.

**Proposition 1.** *The ACH equation (20) admits a pseudo-potential $\gamma$ determined by the compatible equations*

$$\gamma_y = -\frac{1}{2\lambda p}\gamma^2 + \frac{p}{2} + \frac{\lambda}{2p}, \qquad \gamma_T = \frac{\gamma^2}{2} + \frac{p_T}{p}\gamma + \lambda u - \frac{1}{2}\lambda^2. \tag{21}$$

*Moreover, the following two systems of equations are compatible whenever $p(y, T)$ and $u(y, T)$ satisfy (20):*

$$\delta_y = \frac{\gamma}{p}, \qquad \delta_T = \lambda\left(-\frac{p_T}{p} - \gamma\right), \tag{22}$$

*and*

$$\beta_y = \frac{p}{2}e^{\delta/\lambda}, \qquad \beta_T = \frac{1}{2}(-\gamma^2 + \lambda^2)\, e^{\delta/\lambda}. \tag{23}$$

**Theorem 3.** *The first-order generalized symmetries of the augmented associated CH system (20)–(23), represented by vector fields (9), with $G^\alpha$ and $H_b$ being functions of the variables $p, u, \gamma, \delta, \beta, p_y, u_y, u_T$ only, are linear combinations of*

$$W_1 = -p^2 u_y \frac{\partial}{\partial p} + u_T \frac{\partial}{\partial u} - \left(\frac{\lambda^2}{2} - \lambda u - \frac{\gamma^2}{2} + p\gamma u_y\right)\frac{\partial}{\partial \gamma} - \lambda(\gamma - p u_y)\frac{\partial}{\partial \delta} + \frac{1}{2}\,\mathrm{e}^{\delta/\lambda}(\lambda^2 - \gamma^2)\frac{\partial}{\partial \beta}, \tag{24}$$

$$W_2 = p_y \frac{\partial}{\partial p} + u_y \frac{\partial}{\partial u} + \left(\frac{\lambda}{2p} + \frac{p}{2} - \frac{\gamma^2}{2\lambda p}\right)\frac{\partial}{\partial \gamma} + \frac{\gamma}{p}\frac{\partial}{\partial \delta} + \frac{1}{2}\,\mathrm{e}^{\delta/\lambda} p \frac{\partial}{\partial \beta}, \tag{25}$$

$$W_3 = \lambda \frac{\partial}{\partial \delta} + \beta \frac{\partial}{\partial \beta}, \tag{26}$$

$$W_4 = \frac{\partial}{\partial \beta}, \tag{27}$$

$$W_5 = 2\,\mathrm{e}^{\delta/\lambda} p\gamma \frac{\partial}{\partial p} + 2\,\mathrm{e}^{\delta/\lambda}\lambda(\gamma - p u_y)\frac{\partial}{\partial u} - \mathrm{e}^{\delta/\lambda}(\lambda^2 - \gamma^2)\frac{\partial}{\partial \gamma} - 2\lambda(\mathrm{e}^{\delta/\lambda}\gamma - \beta)\frac{\partial}{\partial \delta} + \beta^2 \frac{\partial}{\partial \beta}. \tag{28}$$

*Consequently, these vector fields are nonlocal symmetries of the ACH equation.*

**Corollary 3.** *The five nonlocal symmetries (24)–(28) generate a Lie algebra with the commutator table:*

| | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ |
|---|---|---|---|---|---|
| $W_1$ | | | | | |
| $W_2$ | | | | | |
| $W_3$ | | | | $-W_4$ | $W_5$ |
| $W_4$ | | | $W_4$ | | $2W_3$ |
| $W_5$ | | | $-W_5$ | $-2W_3$ | |

## 4. Applications

The crucial fact about the symmetries appearing in theorems 2 and 3 is that they can be explicitly integrated. Note that symmetries (14), (15), (24) and (25) correspond to translations, while (16), (17), (26) and (27) correspond to 'gauge transformations' of the nonlocal variables. We concentrate therefore in (18) and (28). The flow of the vector field (18) is determined by the following equations [20, 10]:

$$\frac{\partial x}{\partial \tau} = -\,\mathrm{e}^{\delta(\tau,\eta)/\lambda}, \tag{29}$$

$$\frac{\partial m}{\partial \tau} = \frac{2}{\lambda}\gamma(\tau,\eta) m(\tau,\eta)\,\mathrm{e}^{\delta(\tau,\eta)/\lambda}, \tag{30}$$

$$\frac{\partial \gamma}{\partial \tau} = \mathrm{e}^{\delta(\tau,\eta)/\lambda}\left(\frac{1}{2\lambda}\gamma(\tau,\eta)^2 - \frac{1}{2}\lambda\right), \tag{31}$$

$$\frac{\partial \delta}{\partial \tau} = \beta(\tau,\eta) - \gamma(\tau,\eta)\,\mathrm{e}^{\delta(\tau,\eta)/\lambda}, \tag{32}$$

$$\frac{\partial\beta}{\partial\tau} = \frac{1}{2\lambda}\beta(\tau,\eta)^2, \tag{33}$$

in which $\tau$ represents a flow parameter. There is no need to consider an equation for $u(\tau)$: it is proven in [22] that $u(\tau)$ is in fact determined by (29)–(33).

**Proposition 2.** *The initial value problem (29)–(33) with initial conditions $\beta_0 = \beta(0,\eta)$, $\gamma_0 = \gamma(0,\eta)$, $\delta_0 = \delta(0,\eta)$, $m_0 = m(0,\eta)$ and $x_0 = x(0,\eta) = \eta$ has the solution*

$$\beta(\tau,\eta) = \frac{1}{B(\eta)}2\lambda\beta_0, \tag{34}$$

$$\gamma(\tau,\eta) = \frac{1}{B(\eta)}[\gamma_0 B(\eta) + \tau e^{\delta_0/\lambda}A^+(\eta)A^-(\eta)], \tag{35}$$

$$\delta(\tau,\eta) = \lambda\ln\left|\frac{4\lambda^2 e^{\delta_0/\lambda}}{(B(\eta)+\tau e^{\delta_0/\lambda}A^+(\eta))(B(\eta)+\tau e^{\delta_0/\lambda}A^-(\eta))}\right|, \tag{36}$$

$$m(\tau,\eta) = \frac{1}{B(\eta)^4}[B(\eta)+\tau e^{\delta_0/\lambda}A^-(\eta)]^2[B(\eta)+\tau e^{\delta_0/\lambda}A^+(\eta)]^2 m_0, \tag{37}$$

$$x(\tau,\eta) = \eta + \ln\left|\frac{B(\eta)+\tau\, e^{\delta_0/\lambda}A^-(\eta)}{B(\eta)+\tau e^{\delta_0/\lambda}A^+(\eta)}\right|, \tag{38}$$

*in which the functions $B$, $A^+$ and $A^-$ are given by*

$$B(\eta) = -\tau\beta_0 + 2\lambda, \qquad A^+(\eta) = \gamma_0 + \lambda, \qquad A^-(\eta) = \gamma_0 - \lambda. \tag{39}$$

This proposition appears in [20]. We remark that it allows us to construct explicit families of solutions to the interesting CH equation. Indeed, it contains a Darboux transformation. Let us assume that the 'old' independent variables are $\eta$ and $t$. Reasoning as in [24] (see also [21]) we have

**Theorem 4.** *The CH equation (10), understood as an equation for m, is invariant under the transformation $\eta \mapsto x$ and $m(\eta,t) \mapsto \bar{m}(x,t)$, in which*

$$x = x(\eta,t) = \eta + \ln\left[\frac{1-\frac{\lambda}{B}\left(\left(\frac{B_\eta}{m}\right)_\eta - \frac{B_\eta}{m}\right)}{1-\frac{\lambda}{B}\left(\left(\frac{B_\eta}{m}\right)_\eta + \frac{B_\eta}{m}\right)}\right], \tag{40}$$

*and $\bar{m}(x,t)$ is obtained by inverting (40) and replacing into*

$$\bar{m} = \exp\left[2\left(x(\eta,t)-\eta\right)\right]\left[1-\frac{\lambda}{B}\left(\left(\frac{B_\eta}{m}\right)_\eta + \frac{B_\eta}{m}\right)\right]^4 m. \tag{41}$$

*In equations (40) and (41), $B = B(\eta) = -\tau\beta(\eta,t) + 2\lambda$, the functions $m(\eta,t)$ and $\beta(\eta,t)$ are related by*

$$m = \lambda\frac{\partial^2}{\partial\eta^2}\ln\left(\frac{\beta_\eta}{m}\right) + \frac{\lambda}{2}\left[\frac{\partial}{\partial\eta}\ln\left(\frac{\beta_\eta}{m}\right)\right]^2 - \frac{\lambda}{2}, \tag{42}$$

*and $\beta(\eta,t)$ is a solution to the equation obtained from replacing (42) into*

$$\beta_t = \frac{\beta_\eta}{m}\left(-\frac{\lambda^2}{2}\left[\frac{\partial}{\partial\eta}\ln\left(\frac{\beta_\eta}{m}\right)\right]^2 + \frac{\lambda^2}{2} - um\right). \tag{43}$$

In the case of the ACH equation (20), the foregoing theory allows us to obtain two Darboux transforms. First, we consider the pseudo-potential $\gamma(y, T)$ determined by equations (21). Computing $p(y, T)$ from the first equation in (21) and replacing into the equation for $\gamma_T$, we find an equation for $\gamma$. Proceeding as in Chen [5] and Chern and Tenenblat [6], we study the discrete symmetries of this latter equation and we find

**Proposition 3.** *If $\gamma(y, T)$ is determined by equations (21), and $p(y, T)$ is a solution to the ACH equation (20), then so is*

$$\overline{p}(y, T) = p(y, T) - 2\frac{\partial\gamma}{\partial y}(y, T).$$

This transformation was found by Schiff in [23] using loop groups techniques, and it has been re-derived by Hone from a relationship between KdV and ACH found by him in [11]. The fact that it depends *only* on the existence of a quadratic pseudo-potential for ACH is of some interest.

Now, the flow of the vector field (28) with initial conditions $u(y, T, 0) = u_0$, $p(y, T, 0) = p_0$, $\gamma(y, T, 0) = \gamma_0$, $\delta(y, T, 0) = \delta_0$ and $\beta(y, T, 0) = \beta_0$ is

$$\beta(\tau) = \frac{\beta_0}{1 - \tau\beta_0}, \tag{44}$$

$$\gamma(\tau) = \frac{\tau\left(\lambda^2\omega_0 - \omega_0\gamma_0^2 + \gamma_0\beta_0\right) - \gamma_0}{\tau\beta_0 - 1}, \tag{45}$$

$$\delta(\tau) = \delta_0 - \lambda\ln[(1 - \tau\beta_0 + \omega_0\gamma_0\tau - \tau\lambda\omega_0)(1 - \tau\beta_0 + \omega_0\gamma_0\tau + \tau\lambda\omega_0)], \tag{46}$$

$$p(\tau) = \frac{p_0(1 - \tau\beta_0 + \omega_0\gamma_0\tau - \tau\lambda\omega_0)(1 - \tau\beta_0 + \omega_0\gamma_0\tau + \tau\lambda\omega_0)}{(-1 + \tau\beta_0)^2}, \tag{47}$$

in which $\omega_0 = \exp(\delta_0/\lambda)$. As in the CH equation case, we do not calculate explicitly $u(\tau)$, since this function is completely determined by $p(\tau)$, $\gamma(\tau)$, $\delta(\tau)$ and $\beta(\tau)$; see [10]. These formulae yield whole families of non-trivial solutions to the ACH equation (cf Hone [11] for another construction of solutions to ACH). Moreover, they imply

**Proposition 4.** *Assume that $p(y, T)$ solves the ACH equation (20), and that $\beta(y, T)$ is a solution to*

$$\frac{\beta_T}{\beta_y} = \frac{-\gamma^2 + \lambda^2}{p}, \tag{48}$$

*in which $\gamma$ is a solution to the compatible system (21). Then, the function $\overline{p}(y, T)$ given by*

$$\overline{p} = p\left(-1 + \frac{2\tau\lambda p}{-1 + \tau\beta}\left(\frac{\beta_y}{p}\right)_y\right)^2 - \frac{4\lambda^2\tau^2}{(-1 + \tau\beta)^2}\frac{\beta_y^2}{p} \tag{49}$$

*also solves (20).*

We note that the first equation in (21) implies that equation (48) can be written as

$$\frac{\beta_T}{\beta_y} = -\lambda(p - 2\gamma_y),$$

and that proposition 3 says that $\hat{p} = p - 2\gamma_y$ is a new solution to the ACH equation. We therefore interpret proposition 4 as providing us with a nonlinear superposition rule for ACH.

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*Note added in proof.* We have found infinite Lie algebras of nonlocal symmetries containing the Lie algebras of Corollaries 2 and 3. Their constructions are detailed in [10].

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